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A PROBLEM IN THE THEORY OF MIXING LAYERS†

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The classical problem of the free steady mixing layer which is formed as the result of the interaction between two parallel homogeneous flows which move with different velocities and come into contact in a certain section is considered. Subject to the additional condition that the first derivative of the solution in a class of self-similar functions is positive, a boundary-value problem is studied, for values of the self-similarity index $m > 0$, which describes the mixing of two viscous streams of the same fluid for $m = 1$ [1] and for $m = 2$ [2]. The method of investigation used [3–5] enables the third-order non-linear equation to be reduced to a first-order equation and enables the corresponding solutions $\Phi(\zeta)$ to be constructed in a parametric form as a function of the values of m. A knowledge of the behaviour of the velocity profile of the main stream can be used to investigate the flow stability. The results obtained form the basis of the subsequent construction of the solution of Lock's problem [6] and the investigation of the uniqueness of the solutions obtained. © 1997 Elsevier Science Ltd. All rights reserved.

It is well known [1, 7–9] that the flow in a mixing layer is described by the boundary-layer equation for the stream function with a zero pressure gradient. The latter equation has a class of self-similar solutions. The self-similar function $\Phi(\zeta)$ (ζ is the self-similar variable) satisfies a well-known non-linear differential equation, the coefficients of which contain the self-similarity index m [3–5].

A solution of the boundary-value problem which describes the flow in the mixing layer is sought in the class of self-similar functions when $m = 1$ [1, 3–5]. The problem for $m = 1$ was subsequently extended [6] to the case of the mixing of two plane-parallel streams of fluid with different densities and coefficients of viscosity. Experiments [8] show that the mixing layer at the interface between two different media often turns out to be more stable than a free mixing layer in a homogeneous fluid.

Non-classical problems for mixing layers with values of $m \in (1, 2]$ have also arisen [2, 10–12] in relation to the development of the theory of free interaction. The flow in the neighbourhood of the trailing edge of a plate when two viscous streams of the same fluid leave the edge at different velocities was investigated in [2]. The flow in the asymmetric Goldstein wake which arises is described by a boundary-value problem which must be satisfied by a self-similar function $\Phi(\zeta)$ for $m = 2$.

In all of the above-mentioned papers the investigations were carried out by numerical and asymptotic methods. The question regarding the existence and uniqueness of a solution was left open.

1. The flow of an incompressible fluid in mixing layers is described in the first approximation by a boundary-layer equation with a zero pressure gradient in the stream function Ψ

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=\frac{\partial^3\psi}{\partial y^3} \tag{1.1}$$

Equation (1.1) has solutions in the class of self-similar functions [7–9]

$$\psi=\left[\frac{m}{m+1}\right]^{-1/2}x^{\frac{m}{m+1}}\Phi(\zeta),\ \ \zeta=\left[\frac{m}{m+1}\right]^{1/2}yx^{-\frac{1}{m+1}}$$

A third-order non-linear differential equation

$$\frac{m-1}{m}\left(\frac{d\Phi}{d\zeta}\right)^2-\Phi\frac{d^2\Phi}{d\zeta^2}=\frac{d^3\Phi}{d\zeta^3} \tag{1.2}$$

is obtained for determining $\Phi(\zeta)$.

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The boundary conditions for Eq. (1.2) are formulated as follows [3–5]:

$$\Phi = b_1\zeta^m + \ldots, \ \zeta \to +\infty, \ b_1 > 0, \ m > 0 \tag{1.3}$$

$$\Phi = 0, \ \zeta = 0 \tag{1.4}$$

$$\Phi = b_2(-\zeta)^m + \ldots, \ \zeta \to -\infty, \ b_2 < 0 \tag{1.5}$$

A boundary-value problem of the type of (1.2), (1.3)–(1.5) has been investigated in [1, 2] for $m = 1$ and $m = 2$. In the physical problems which it describes, the x axis of an orthogonal system of coordinates (x, y) coincides with the zero streamline. The boundary-value problem (1.2), (1.3)–(1.5) is also of mathematical interest in its own right. Equation (1.2) is invariant under a displacement transformation. Its order is therefore reduced if we

$$f = \frac{d\Phi}{d\zeta}\left(\frac{d^2\Phi}{d\zeta^2} = f\frac{df}{d\zeta}, \ \frac{d^3\Phi}{d\zeta^3} = f\left[\left(\frac{df}{d\zeta}\right)^2 + f\frac{d^2f}{d\xi^2}\right], \ \xi \equiv \Phi\right)$$

A boundary-value problem

$$f\frac{d^2f}{d\xi^2} + \left(\frac{df}{d\xi}\right)^2 + \xi\frac{df}{d\xi} - \frac{m-1}{m}f = 0 \tag{1.6}$$

$$f = mb_1^{\frac{1}{m}}\xi^{\frac{m-1}{m}} + \ldots, \ \xi \to +\infty; \ f = m(-b_2)^{\frac{1}{m}}(-\xi)^{\frac{m-1}{m}} + \ldots, \ \xi \to -\infty$$

is obtained for determining $f(\xi)$.

Equation (1.6), in turn, is invariant under the stretching transformation $f \to \alpha^2 f$, $\xi \to \alpha\xi$, $\alpha \neq 0$. This means the order of Eq. (1.6) is reduced by making the substitution

$$f = \xi^2 F(\xi), \ \xi dF/d\xi = \Psi \tag{1.7}$$

As a result, Eq. (1.2) reduces to the first-order equation [3–5]

$$\frac{d\Psi}{dF} = -\frac{\Psi^2 + 7F\Psi + 6F^2 + \Psi + \dfrac{m+1}{m}F}{F\Psi} = -\frac{P(F,\Psi)}{F\Psi} \tag{1.8}$$

The derivatives of the functions $f(\xi)$ and $\Phi(\zeta)$ are connected with F and Ψ by the relations

$$\begin{gathered}\frac{d\Phi}{d\zeta} = \xi^2 F, \ \frac{d^2\Phi}{d\zeta^2} = \xi^3 F(\Psi + 2F), \ \frac{d^3\Phi}{d\zeta^3} = -\xi^4 F(\Psi + \frac{m+1}{m}F) \\ \frac{df}{d\xi} = \xi(\Psi + 2F), \ \frac{d^2f}{d\xi^2} = -\frac{(\Psi + 2F)^2 + \Psi + \dfrac{m+1}{m}F}{F} = -\frac{R(F,\Psi)}{F}\end{gathered} \tag{1.9}$$

The equations $P = 0$ and $R = 0$ determine the curves P_1 and R_1 which pass through the point B and the curves P_2 and R_2 which pass through the point A (Fig. 1).

As a result of the subsequent group transformations, the problem of investigating the integral curves (ICs) of Eq. (1.2) was reduced to a problem involving the study of the pattern of the ICs of the first-order equation (1.8). In order to construct this pattern, it is necessary to know the nature of the singular points of Eq. (1.8) and to find the requirements which the ICs must satisfy in order that the boundary conditions are satisfied.

Equation (1.8) has three singular points $A(0, 0)$, $B(0, -1)$, $C(-(m + 1)/6m, 0)$ in the finite part of the plane and three singular points at infinity. These are conveniently denoted by $E - E_*$; $G - G_*$, $Q - Q_*$ since each of these singular points on the equator of a Poincaré unit hemisphere is split into two identical ones which lie symmetrically about the centre. Those which the curves enter or from which the curves

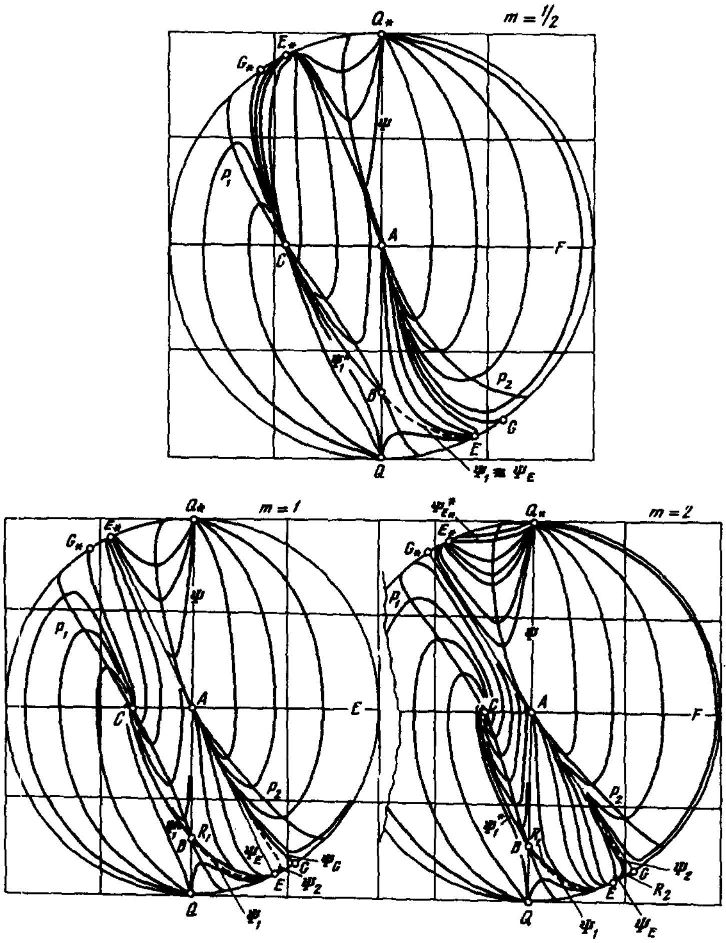

Fig. 1.

leave are denoted by E, G, Q when $\Psi < 0$ and by E_*, G_*, Q_* when $\Psi > 0$. The hemisphere is then projected onto the unit circle (Fig. 1) on which the singular points $E - E_*$; $G - G_*$, $Q - Q_*$ and $Q - Q_*$ have the coordinates $(\pm 1/\sqrt{(5)} \mp 2/\sqrt{(5)})$, $(\pm 2/\sqrt{(3)}, \mp 3/\sqrt{(13)})$ and $(0, \mp 1)$, respectively. The integral curves of Eq. (1.8) can only traverse the $F = 0$ axis by way of the singular points A, B and $Q - Q_*$.

We shall now study the behaviour of the ICs of Eq. (1.8) in the neighbourhood of the singular points.

The singular point $A(0, 0)$. The singular point A is characterized by the fact that one of the characteristic numbers of Eq. (1.8) is equal to zero while the other is non-zero. We denote the domains in which $d\Psi/dF < 0$ by $\Omega = \{F > 0, P(F, \Psi) < 0\}$, $\Omega_* = \{F < 0, \Psi > 0, P(F, \Psi) < 0\}$. The IC which hit the point A either belong to Ω or to Ω^* in a certain neighbourhood of this point [13–16]. If $(F, \Psi) \in \Omega$, then a single IC, which we denote by Ψ^*_A, hits A. If an IC in domain Ω enters a certain neighbourhood of point A, then it necessarily hits the point A when $F \to +0$. We now consider the arc KL of radius ρ $(0 < \rho < 1)$, the point K of which lies on the negative part of the Ψ axis while L lies on the curve P_2. The curvilinear sector ALK, the boundary of which is the segment AK, the arc KL and a part of the curve P_2–AL contain a single critical direction [17, 18].

Using the technique developed in [15–18], it can be shown that all the ICs which enter the sector ALK hit the point A when $F \to +0$, touching the line $\Psi = -[(m+1)/m]F$. We make the substitution $\Psi = [-(m+1)/m + \omega]F$. For the function $\omega(F)$, we obtain the equation

$$F^2 \frac{d\omega}{dF} = \left[\omega + \frac{(m-1)(m-2)}{m^2} F + \frac{3m-4}{m} F\omega + 2\omega^2 F\right] \left(\frac{m+1}{m} - \omega\right)^{-1} \tag{1.10}$$

It follows from the above that $\omega(F) \to 0$ when $F \to +0$. In the neighbourhood of the point $(0, 0)$, Eq. (1.10) is Bendixon's equation [13, 16]. Hence a neighbourhood of the point (0, 0) exists, through which all the ICs of Eq. (18) from the sector ALK pass and reach (0, 0) on coming into contact with the line $\omega = -[(m-1)(m-2)/m^2]F$. This means that any IC which is contained in the sector ALK, has the form $\omega = [(m-1)(m-2)/m^2]F + \omega_1(F)\,F$ when $F \to +0$. The function $\omega_1(F) \to 0$, when $F \to +0$, again satisfies Bendixon's equation in a certain neighbourhood of the point (0, 0).

Consequently, for any IC which hits the point A at $F \to +0$ for a certain $\delta_0 > 0$ (depending on the IC), the representation

$$\Psi = -\frac{m+1}{m}F - \frac{(m-1)(m-2)}{m^2}F^2 + \omega_2(F),\quad \omega_2(F) = O(F^3) \tag{1.11}$$

$$F \in [0, \delta_0]$$

holds.

We now distinguish one of the integral curves $\Psi = \mu(F)$ and investigate how the remaining integral curves of Eq. (1.8) differ from it when $F \to +0$. We will seek a solution of Eq. (1.8) in the form $\Psi = \mu + \nu$. Substitution into Eq. (1.8) gives

$$E\mu\frac{dv}{dF} = -\left(1 + 7F + 2\mu + F\frac{d\mu}{dF}\right)v - v^2 - Fv\frac{dv}{dF} \tag{1.12}$$

The solution of the truncated equation (1.12), obtained by neglecting the non-linear terms, has the form

$$v = Cv_0 = C\omega_3(F)F^l \exp\left[-\frac{m}{m+1}F^{-1}\right],\quad \omega^3(F) = 1 + O(F) \tag{1.13}$$

$$l = \frac{3m^2 + 4m - 5}{m(m+1)},\quad C = \text{const},\quad F \in (0, \delta_0]$$

All the functions of the form $F^q v_0(F)$ (the magnitude of q is arbitrary) which are encountered below are supplemented by zero with respect to their continuity at zero. We put $\nu = C\,(F)\nu_0$ and, for determining $C(F)$, we obtain the integral equation

$$C(F) = A(C; D)$$

$$A(C;D) = -\int_0^F F^{-1}\mu^{-1}\frac{d(Fv_0)}{dF}\frac{C^2}{1+\mu^{-1}v_0C}dF + D,\quad D = \text{const} \tag{1.14}$$

We now consider a space of continuous functions in the segment $[0, \delta]$ with a metric $\|\varphi\| = \sup|\varphi|$, $F \in [0, \delta]$ and denote the bounded and closed set of functions which satisfy the conditions

$$\|\varphi(F) - D\| \leqslant M,\quad \varphi \in C[0,\delta],\quad M = \text{const}$$

by L.

We shall show that the operator $A\,(\varphi); D)$ contracts L. We choose $\delta > 0$ such that relations (1.11), (1.13) and

$$\int_0^\delta \left|F^{-1}\mu^{-1}\frac{d(Fv_0)}{dF}\right|dF \leqslant \frac{Mq_1^2(1-q_2)^2}{(|D|+M)^2},\quad 0 < q_1^2 < \frac{1}{3}$$

$$|\mu^{-1}|v_0(|D| + M) \leqslant q_1 < 1$$

are simultaneously satisfied.

We then have

$$\varphi \in L \to A(\varphi; D) \in C[0,\delta],\quad \|A(\varphi, D) - D\| \leqslant M$$

$$\varphi_1, \varphi_2 \in L \to \|A(\varphi_1; D) - A(\varphi_2; D)\| \leqslant \theta^2\|\varphi_1 - \varphi_2\|,\quad \theta^2 < 1$$

It follows from this that Eq. (1.14) has a unique solution $C^*(F) \in L$, $F \in [0, \delta]$. Consequently, any solution of Eq. (1.8) in the domain Ω when $F \to +0$ can be represented in the form

$$\Psi = \mu(F) + Dv_0 - v_0 \int_0^F \tau^{-1}\mu^{-1} \frac{d(\tau v_0)}{d\tau} \frac{C^{*2}}{1+\mu^{-1}v_0 C^*} dF = \mu(F) + Dv_0 + o(v_0) \tag{1.15}$$

It will become clear from what follows that it is best to take the IC Ψ_E, which will be determined below, to serve as $\Psi = \mu(F)$.

The asymptotic behaviour of the solutions of Eq. (1.8) at $\zeta \to +\infty$, which correspond to (1.5), will be as follows [3–5]:

$$\Phi = b_1(\zeta + l_A)^m - \frac{(m-1)(m-2)}{m+1} O\left[(\zeta + l_A)^{-(m+2)}\right] + \tag{1.16}$$

$$+D_2(\zeta + l_A)^{\beta_1} \exp\left[-\frac{b_1}{m+1}(\zeta + l_A)^{m+1}\right], \ \beta_1 = -\frac{3m^2+5m-4}{m+1}; \ D_2, l_A = \text{const}$$

The singular point B. The singular point B is a saddle point. For any $m > 0$, a single integral curve passes through it which, at $F \to 0$, is described by the expansion [3–5]

$$\Psi = -1 + \sum_{k=1}^{\infty} b_k F^k, \ b_1 = -\frac{6m-1}{2m}, \ b_2 = \frac{1}{3}\left[2b_1^2 + 7b_1 + 6\right] \tag{1.17}$$

$$b_k = \frac{1}{k+1}\left[\frac{k+2}{2} \sum_{n=2}^{k} b_{k-1} b_{k-n+1} + 7b_{k-1}\right], \ k \geq 3$$

We will denote the part of it when $F > 0$ by Ψ_1 and the part of it when $F \leq 0$ by Ψ_1^*. The solutions of Eq. (1.6) which correspond to expansion (1.17) can be represented in the form

$$f = -c(\xi - c) - \frac{1}{4m}(\xi - c)^2 + O[(\xi - c)^3], \ \xi \to c \neq 0 \tag{1.18}$$

Solutions which are described by expansion (1.18) when $f > 0$ correspond to the curve Ψ_1 and, when $f < 0$, to the curve Ψ_1^*. The solutions of Eq. (1.2) which correspond to Ψ_1 and Ψ_1^* at $F \to 0$ behave in the following manner at $c\zeta \to +\infty$

$$\Phi = c - \text{sign}(cf)\exp\left[-c(\zeta + l_B)\right] + \ldots, \ l_B = \text{const} \tag{1.19}$$

Henceforth, ζ with a subscript denotes a certain finite value of ζ, and C with a subscript denotes a constant.

The singular point $E - E_*$. In order to investigate a singular point at infinity, we make the change of variables

$$2F - \frac{m-1}{m} = \frac{1}{t}, \ \Psi = \frac{\sigma - 1}{t}$$

in Eq. (1.8).

As a result, we obtain the equation

$$t(1-\sigma)(1+\frac{m-1}{m}t)\frac{d\sigma}{dt} = \frac{\sigma}{2} - 2\sigma^2 - \frac{5m-3}{2m}\sigma t - \tag{1.20}$$

$$-\frac{(2m-1)(m-1)}{m^2}t^2 - \frac{m-1}{m}t\sigma^2$$

Its solution, when $0 \leq t < r\,(C_E)$, can be represented in the form of a convergent series (r is the radius of convergence) [3–5, 19]

$$\sigma = \sum_{k=1}^{\infty} d_k t^{k/2} = C_E t^{1/2} J_1(t; C_E) + J_2(t; C_E)$$
$$J_1(t, C_E) = J_1(t; -C_E), \quad J_2(t; C_E) = J_2(t; -C_E), \quad 0 \leqslant t < r(C_E) \tag{1.21}$$

The solution when $t \to 0$ is constructed in a similar manner. It follows from what has been said that the point $E - E_*$ is a node. According to (1.21), the solution of Eq. (1.8) in the neighbourhood of the point E behave in the following manner at $F \to +\infty$

$$\Psi = -2F + \frac{m-1}{m} + C_E\sqrt{2F - \frac{m-1}{m}} + \ldots, \quad F \to +\infty \tag{1.22}$$

We will now investigate how F depends on ξ in the neighbourhood of the point E. A new function τ is introduced by putting $t = \tau^2$. To determine $\tau - \tau(\xi)$ we obtain an equation from (1.7), the general integral of which will be

$$\Phi(\xi, \tau; a, C_E) = a\xi - \tau \exp\left[\int_0^{\tau} \frac{z(\theta; C_E) d\theta}{\theta(1 - z(\theta; C_E))}\right] = 0, \quad a \neq 0$$
$$z(\tau; C_E) = C_E \tau J_1(\tau^2; C_E) + J_2(\tau^2; C_E) \tag{1.23}$$

Since $z(\theta; C_E) = z(-\theta; -C_E)$, we have

$$\Phi(\xi, -\tau; -a, -C_E) = -\Phi(\xi, \tau; a, C_E)$$

We will denote the ICs which, when $F \to +\infty$, are described by expansions (1.22) with $C_E = \alpha > 0$ by $\Psi = \Psi(F; \alpha)$ and with $C_E = \beta < 0$ by $\Psi = \Psi(F; \beta)$. The functions $z(\tau; C_E)$ corresponding to them are denoted by $z(\tau; \alpha)$ and $z(\tau; \beta)$.

If we find ourselves on the curve $\Psi = \Psi(F; \alpha)$ and $a = a_2 > 0$ then Eq. (1.23) serves for determining the dependence of τ on ξ, and $\tau_2 = \xi\varkappa(\xi; a_2, \alpha)$, $|\xi| < r_\varepsilon(a_2, \alpha)$ will be its solution, where κ is a holomorphic function and $\kappa(0; a_2, \alpha) = a_2$.

Suppose we find ourselves on the curve $\Psi = \Psi(F; \beta)$ and $a = a_1 < 0$. We find how τ depends on ξ as the solution of Eq. (1.23) and we denote this solution by $\tau_1 = \xi\varkappa(\xi; a_1, \beta)$, $|\xi| < r_E(a_1, \beta)$, $\varkappa(0; a_1, \beta) = a_1$.

It follows from the continuity of $f(\xi)$ and its derivative at the point ξ that $a_1^2 = a_2^2$ and $a_2 = \beta a_1$. Since α and β must have different signs, the solution of this system will be $\beta = -\alpha$, $a_1 = -a_2$.

On putting $\beta = -\alpha$, $a_1 = -a_2 = -a_0 < 0$, we obtain from relation (1.23) that $\tau_1(\xi; -a_0, -\alpha) = -\tau_2(\xi, a_0, \alpha)$, $|\xi| < r_E(a_0, \alpha)$.

Using (1.7), we therefore have

$$f_1 = \frac{1}{2}\left[\frac{m-1}{m}\xi^2 + \varkappa^{-2}(\xi; -a_0, -\alpha)\right], \quad f_2 = \frac{1}{2}\left[\frac{m-1}{m}\xi^2 + \right.$$
$$\left. + \varkappa^{-2}(\xi; a_0, \alpha)\right], \quad f_1 \equiv f_2, \quad 0 \leqslant |\xi| < r_E(a_0, \alpha) \tag{1.24}$$

Consequently, each doubly continuously differentiable solution $f(\xi)$ in the neighbourhood of the point ($\xi = 0, f(0) \neq 0$) is mapped, in the neighbourhood of point E, onto the two ICs $\Psi = \Psi(F; -\alpha)$, $\Psi = \Psi(F; \alpha)$, the combination of which we shall also call an IC. Conversely, if the ICs $\Psi = \Psi(F; -\alpha)$ and $\Psi = \Psi(F; \alpha)$ are extended, the solutions corresponding to them in the neighbourhood of the points ($\xi = 0, f(0) \neq 0$) in the class of doubly continuously differentiable functions will be described by expansions (1.24). The solutions of Eq. (1.2) corresponding to them in the (ζ, Φ) plane have the form

$$\Phi = \frac{a_0^{-2}}{2}(\zeta - \zeta_E) + \frac{a_0^{-3}}{4}\alpha(\zeta - \zeta_E)^2 + \frac{m-1}{24m}a_0^{-4}(\zeta - \zeta_E)^3 + \ldots \tag{1.25}$$

Conversely, each solution $\Phi(\zeta)$ of Eq. (1.2) in the neighbourhood of the point $\zeta = \zeta_E$, $\Phi(\zeta_E) = 0$ with $d\Phi/d\zeta_E > 0$ is mapped into the corresponding neighbourhood of point E by the two branches of the ICs of Eq. (1.8) with $C_E = \alpha > 0$ and $C_E = \beta = -\alpha < 0$.

If $\beta \neq -\alpha$ and the first derivative of $\Phi'(\zeta)$ is continuous, on passing through point E in the manner described above, the second derivative $\Phi''(\zeta)$ will have a discontinuity at the point $\zeta = \zeta_E$.

On passing through point E in the manner described above, the solutions of Eq. (1.2) in the neighbourhood of points of the ζ axis behave as follows:

$$\Phi = -\frac{a_0^{-2}}{2}\left(\zeta - \zeta_{E_*}\right) + \frac{a_0^{-3}}{4} C_{E_*}\left(\zeta - \zeta_{E_*}\right)^2 + \frac{m-1}{24m} a_0^{-4}(\zeta - \zeta_{E_*})^3 + \ldots \tag{1.26}$$

We will denote the IC which departs from E when $F \to +\infty$ and is described by (1.22) with $C_E = 0$ by Ψ_E. Each of the solutions of Eq. (1.2) corresponding to it is described by expansion (1.25) with $\alpha = 0$.

The IC Ψ_{E_*} is defined in a similar manner. Curves which are described by (1.26) with $C_{E_*} = 0$ correspond to it in the (ζ, Φ) plane.

The singular point $Q - Q_*$. The singular point $Q - Q_*$ is a node. In the neighbourhood of this point, the ICs behave as follows:

$$\Psi = \frac{C_Q}{F} - 1 + \ldots, \quad F \to 0, \quad C_Q \neq 0$$

In the (ζ, Φ) plane, the point $Q - Q_*$ corresponds to points of a local extremum in the solutions of Eq. (1.2). In the neighbourhood of the extremal point $\zeta = \zeta_Q$, the solution can be represented in the form

$$\Phi = c + \frac{C_Q c^3}{2}\left(\zeta - \zeta_Q\right)^2 - \frac{C_Q c^4}{6}\left(\zeta - \zeta_Q\right)^3 + \ldots$$

It is clear from this expansion that, on passing through the point $Q - Q_*$, the constant C_Q must be preserved. Suppose, for example, that an IC hits the point Q with $C_Q < 0$ at $F \to +0$. In order to obtain the solution $\Phi(\zeta)$ in the neighbourhood of the point $\zeta = \zeta_Q$, it is necessary to leave the point Q_* along the IC with $C_{Q_*} = C_Q$ at $F \to -0$.

The singular point $G - G_*$. The singular point $G - G_*$ is a saddle point. A unique IC

$$\Psi_G = -\frac{3}{2}F + \frac{m-2}{5m} + O(F^{-1}), \quad F \to +\infty, \quad m > 0$$

hits this point at $F \to +\infty$.

Integral curves, which are identically not equal to zero and touch the ζ axis, correspond to it in the (ζ, Φ) plane [11, 12].

The singular point C. When $m > m_* = (-17 + 12\sqrt{6})/23 > 1/2$, the singular point C is a focus and, when $0 < m \leqslant m_*$, it is a node. As the IC $\Psi = \Psi(F)$ approaches the singular point C, the ICs corresponding to it in the (ζ, Φ) plane at $\zeta \to \zeta_c$ behave as follows:

$$\Phi = \frac{6m}{m+1}(\zeta - \zeta_C)^{-1} + o\left[(\zeta - \zeta_C)^{-1}\right]$$

2. Hence, if an IC $\Psi = \Psi(F)$, at $F \to +0$, hits point A at $\xi > 0$, then condition (1.3) is satisfied. If it hits point A when $\xi < 0$, then condition (1.5) is satisfied. The transfer at the point $E - E_*$ from the IC $\Psi = \Psi(F; C_E)$ with $C_E = \alpha > 0$ to the IC $\Psi = \Psi(F; -\alpha)$ means that the ICs of Eq. (1.2) corresponding to it intersect the ζ axis.

If the point $E - E_*$ is reached along Ψ_E or Ψ_{E_*}, then $\Phi(\zeta)$ and $\Phi''(\zeta)$ simultaneously vanish. The passage of the singular point $Q - Q_*$ in the manner described above shows that the solutions of Eq. (1.2) corresponding to the IC $\Psi = \Psi(F)$ have a local extremum. As the IC $\Psi = \Psi(F)$ approaches the point C, the ICs of Eq. (1.2) which correspond to it behave as $Q[(\zeta - \zeta_c)^{-1}$.

It has been proved [3–5] that, when $m > 1/2$, the curve Ψ_1 hits the point E with a definite value $C_E = \beta^* < 0$. We now continue this curve with the curve Ψ_2 with $C_E = -\beta^* = \alpha^* > 0$. The curve Ψ_2 hits the point A at $F \to +0$. It has also been shown that, when $m > 1/2$ and at $F \to +0$, Ψ_G and Ψ_E also reach the point A (Fig. 1).

We now consider the boundary condition

$$d\Phi / d\zeta \to 0, \quad \zeta \to -\infty \tag{2.1}$$

Boundary-value problem (1.2)–(1.4), (2.1) gives the solution of the Chapman problem [20, 21] when $m = 1$, and, when $m = 5/3, 2$, it describes the flow in a mixing layer which occurs in the theory of local separation [10–12].

The solution of boundary-value problem (1.2)–(1.4), (2.1) is mapped onto the curve $K = \Psi_1 \cup \Psi_2$ [3, 5]. The curve K will play a cardinal part in the subsequent investigation. We will therefore consider the solution of problem (1.2)–(1.4), (2.1) in greater detail and find its solution in parametric form, assuming the behaviour of curve K to be known. In the (ξ, f) plane, the parametrically defined ICs

$$\Psi_1: \xi = \frac{c_1}{\sqrt{F+1}} \exp \Delta_1, \quad f_1 = \frac{c_1^2 F}{F+1} \exp(2\Delta_1)$$

$$\Delta_1(F) = \int_0^F \left[\frac{1}{\Psi_1} + \frac{1}{2(F+1)}\right] dF, \quad 0 \leq F < +\infty$$

$$\Psi_2: \xi = \frac{c_2}{\sqrt{F}} \exp(-\Delta_2), \quad f_2 = c_2^2 \exp(-2\Delta_2)$$

$$\Delta_2(F) = \int_F^\infty \left[\frac{1}{\Psi_2} + \frac{1}{2F}\right] dF, \quad 0 < F < +\infty$$

correspond to the curves Ψ_1 and Ψ_2.

We now put $c_1 < 0$ and $c_2 = -c_1 \exp[\Delta_1(\infty)]$. Then the function which is the inverse of the function

$$\zeta = \int_0^\Phi \frac{d\xi}{f(\xi)}, \quad f(\xi) = \begin{cases} f_1(\xi), & \xi \in (c_1, 0) \\ c_1^2 \exp[2\Delta_1(\infty)], & \xi = 0 \\ f_2(\xi), & \xi \in (0, \infty) \end{cases}$$

gives the solution of problem (1.2)–(1.4), (2.1) when $m > 1/2$. The values of the constants c_1 and b_1 are related to one another

$$c_1 = -m^{\frac{m}{m+1}} b_1^{\frac{1}{m+1}} \exp\left\{-\Delta_1(\infty) + \int_0^1 \left[\frac{1}{\Psi_2} + \frac{m}{(m+1)F}\right] dF + \Delta_2(1)\right\} = -k_1(m) b_1^{\frac{1}{m+1}} \tag{2.2}$$

It follows from what has been said that a solution of problem (1.2)–(1.4), (2.1) exists when $m > 1/2$ and is unique, as is clear from condition (1.5) of problem (1.2)–(1.5) when $m > 1$. In order to distinguish the unique solution when $m > 1/2$, one can require that $\Phi \to c + O[\exp(-c\,\zeta)]$, $c < 0$ when $\zeta \to -\infty$ instead of condition (2.1). However, the constant c is related to the constant b_1 by relation (2.2) and cannot be assigned arbitrarily. For each m, the coefficient k_1 in (2.2) is calculated once and for all. Since the singular point B is a saddle point, the conditions $\Phi \to c < 0$ and $d\Phi/d\zeta > 0$, $\zeta \in (-\infty, +\infty)$ are sufficient to distinguish the unique solution when $m > 1/2$ [4, 22]. If c is taken as being equal to c_1, we obtain the solution of problem (1.2)–(1.4), (2.1). In order not to introduce a constant into the boundary conditions which has to be determined when solving the problem, the boundary condition $\zeta \to -\infty$ can be formulated as follows: $(-\zeta)^{1+k2}\, \Phi'(\zeta) \to 0$, $\zeta \to -\infty$ for any k.

All the curves situated between Ψ_E and Ψ_1 in the neighbourhood of point A hit point E at $F \to +\infty$, and the curves continued in the manner described above at $F \to +0$ return to point A [4, 5]. The curves situated in the neighbourhood of point A between Ψ_2 and Ψ_G hit point E at $F \to +\infty$, and the continued curves, when $F \to +0$, then hit the point Q.

To prove this, an arc TD, linking the curves P_1 and Ψ_1, is drawn in a certain small neighbourhood of the point B. We take the IC $\Psi = \Psi(F)$ which leaves point E and, in a certain neighbourhood of this point, is described by expansion (1.21) or (1.22) when $C_E = \beta > \beta^*$. We now take a certain point (F_*, Ψ_*) on the curve $\Psi = \Psi(F, \beta)$ in the neighbourhood of the point E and draw the straight line $F = F_*$. This line intersects the curves Ψ_1 and P_1 at the points N and M. We now consider the domain Ω_1, the boundary of which is formed by the arc TD, a part of the curve P_1–TM, a part of the curve Ψ_1–DN and a segment of the straight line $F = F^* - MN$. The inclination of the IC is negative in the domain Ω_1.

When the IC is continued, it can only intersect the boundary of the domain Ω_1 at the points TD or TM. If the IC intersects TD, then, subject to the condition that the radius of the arc TD is small, it leaves the sector BTD across its side BT [15–17]. Hence, when the IC $\Psi = \Psi(F)$ is continued, it necessarily intersects the curve P_1 at a certain point (F_0, Ψ_0). We shall take the curve

$$\Psi_\gamma = -\left(C_\gamma F^{-2} - 3F^2 - \frac{2}{3}\frac{m+1}{m}F\right)^{1/2}, \quad F \in (0, F_0]$$

The constant C_γ is chosen such that the curve Ψ_γ passes through the point (F_0, Ψ_0). At $F \to +0$ the curve Ψ_γ hits the point Q. It can be shown that the inequalities

$$0 < \frac{d\Psi}{dF}(F, \Psi_\gamma) < \frac{d\Psi_\gamma}{dF}$$

hold at the points of the curve Ψ_γ.

It follows from these inequalities that any IC which departs from a point E below the curve Ψ_1 reaches the point Q at $F \to +0$. When continued through the point $Q - Q_*$, it enters a domain in which $F < 0$.

The integral curves which depart from point A above Ψ_G initially intersect the axis F. They then arrive at point Q_* when $F \to +0$ and then, when continued, hit the point C.

3. We now consider problem (1.2)–(1.5) when $m > 1/2$ [4, 5] and the additional condition $d\Phi/d\xi > 0$, $\zeta \in (-\infty, +\infty)$. We shall denote a curve which departs from point A between the curves Ψ_E and Ψ_2 by Ψ_b and a curve between Ψ_E and the Ψ axis by Ψ_H and require that the curves Ψ_b and Ψ_H should be a continuation of one another in the manner described above through the point E. In the (ξ, f) plane, the ICs Ψ_b will correspond to the ICs

$$\xi = \frac{c_b}{\sqrt{F}}\exp[-\Delta_b(F)], \quad f_b = c_b^2 \exp[-2\Delta_b(F)], \quad c_b = \text{const}$$

$$\Delta_b = \int_F^\infty \left(\frac{1}{\Psi_b} + \frac{1}{2F}\right) dF, \quad F \in (0, \infty)$$

These formulae will also describe ICs in the (ξ, f) plane which correspond to Ψ_H if we substitute Ψ_H and c_H (c_H = const) into them instead of Ψ_b and c_b.

We put $c_b = -c_H$. It can then be shown that the function

$$f(\xi) = \begin{cases} f_H(\xi), & \xi \in (-\infty, 0) \\ c_b^2, & \xi = 0 \\ f_b(\xi), & \xi \in (0, +\infty) \end{cases}$$

is doubly continuously differentiable and is a solution of Eq. (1.6). Here the constant $c_b > 0$. Then, at $|\xi| \to \infty$ $(F \to +0)$, we have

$$f_b = \left\{c_b \exp\left[-\int_0^1 \left[\frac{1}{\Psi_b} + \frac{m}{(m+1)F}\right] dF - \int_1^\infty \left[\frac{1}{\Psi_b} + \frac{1}{2F}\right] dF\right]\right\}^{\frac{m+1}{m}} \xi^{\frac{m-1}{m}} + \ldots$$

$$f_H = \left\{-c_H \exp\left[-\int_0^1 \left[\frac{1}{\Psi_H} + \frac{m}{(m+1)F}\right] dF - \int_1^\infty \left[\frac{1}{\Psi_H} + \frac{1}{2F}\right] dF\right]\right\}^{\frac{m+1}{m}} (-\xi)^{\frac{m-1}{m}} + \ldots \tag{3.1}$$

On comparing the asymptotic forms obtained with boundary conditions (1.6), we obtain

$$b_2 = -b_1 \exp\left[-(m+1)\int_0^\infty \left(\frac{1}{\Psi_H} - \frac{1}{\Psi_b}\right) dF\right], \quad c_b > 0 \tag{3.2}$$

When $c_b' < 0$, the function

$$f(\xi) = \begin{cases} f_b(\xi), & \xi \in (-\infty, 0); \\ c_b^2, & \xi = 0; \\ f_H(\xi), & \xi \in (0, +\infty). \end{cases}$$

will also be a solution of Eq. (1.6).

If $c_b < 0$, then

$$b_2 = -b_1 \exp\left[(m+1)\int_0^\infty \left(\frac{1}{\Psi_H} - \frac{1}{\Psi_b}\right) dF\right], \quad c_b < 0 \tag{3.3}$$

Since $\Psi_H^{-1} - \Psi_h^{-1} \geqslant 0$, then, in the case of (3.2), we have $(-b_2) \leqslant b_1$ and, in the case of (3.3), $(-b_2) \geqslant b_1$. The function which is inverse to

$$\zeta = \int_0^\Phi \frac{d\xi}{f(\xi)}$$

is the solution of problem (1.2)–(1.5).

We now consider the integral

$$\int_0^\infty \left[\frac{1}{\Psi_H} - \frac{1}{\Psi_b}\right] dF = \int_0^\infty \left[\frac{1}{\Psi_H} - \frac{1}{\Psi_E}\right] dF + \int_0^\infty \left[\frac{1}{\Psi_E} - \frac{1}{\Psi_b}\right] dF = I_1 + I_2$$

The functions $I_1(\beta), I_2(-\beta), \beta \in (\beta^*, 0]$ are monotonically decreasing and continuous functions. When $\beta = 0$, they are equal to zero. We will now show that the function $I_1(\beta)$ is unbounded. The straight line $\psi = -1$ is drawn. Suppose that the solution $\Psi = \Psi_\varepsilon(F)$ has the asymptotic representation (1.11) in the interval $(0, F_\delta)$ and that the straight line $F = F_\delta$ first intersects Ψ_E and then $\Psi = -1$.

We denote the points of intersection of the straight line $F = F_\delta$ with Ψ_E and the straight line $\Psi = -1$ by L_1 and L_2 respectively, and the domain bounded by the curve Ψ_E and the segments AB, BL_2 and L_1L_2 by Ω_2. We now consider the differential equation

$$\frac{d\Psi}{dF} = -\frac{\Psi_E^2 + 7F\Psi_E + 6F^2 + \Psi + \dfrac{m+1}{m}F}{F\Psi_E} = N(F, \Psi) \tag{3.4}$$

in this domain.

Its solution is $\Psi = \Psi_\varepsilon(F)$. The IC of Eq. (3.4) which departs from any point $(F_0, -1)$, $F_0 \in (0, F_\delta)$ is situated between Ψ_E and the IC of Eq. (1.8) since

$$-\frac{P(F, \Psi)}{F\Psi} > N(F, \Psi), \quad (F, \Psi) \in \Omega_2$$

Equation (3.4) is linear. Its general solution is

$$\Psi = \Psi_E + D_E \exp\left[-\int_{F_0}^F \frac{1+7F}{F\Psi_E} dF\right], \quad D_E = \text{const}$$

The *IC*

$$\tilde{\Psi} = \Psi_E - [1 + \Psi_E(F_0)] F_0^{-\beta_a} \exp\left[\frac{m}{(m+1)F_0}\right] \left[F^{\beta_a} \exp\left(\int_{F_0}^F \omega_2(F) dF\right)\right] \exp\left[-\frac{m}{(m+1)F}\right]$$

passes through the point $(F_0, -1)$.

We now evaluate the integral I_1

$$\int_0^\infty \left[\frac{1}{\Psi_H} - \frac{1}{\Psi_E}\right] dF \geqslant \int_0^{F_0} \left[\frac{1}{\tilde{\Psi}} - \frac{1}{\Psi_E}\right] dF + \int_{F_0}^{F_\delta} \left[-1 - \frac{1}{\Psi_E}\right] dF +$$

$$+ \int_{F_\delta}^\infty \left[\frac{1}{\tilde{\Psi}_H} - \frac{1}{\Psi_E}\right] dF \geqslant \frac{m}{m+1} \ln \frac{F_\delta}{F_0} + O(F_\delta - F_0)$$

Here, $\widetilde{\Psi}_H$ is the IC of Eq. (1.8) which passes through the point $(F_\delta, -1)$. When $F \to F_0$, the quantity $B \to B_*$. Consequently,

$$\lim_{\beta\to\beta^*} I = \infty$$

since the function $I_2(-\beta)$ $(\beta \in (\beta^*, 0])$ is bounded. This means that $(-b_1/b_2) \in (0, \infty)$ and the values of the constants b_1 and b_2 can be chosen arbitrarily.

Hence, if the ratio $(-b_1/b_2)$ is specified, then, from (3.2) and (3.3) we obtain the completely defined values $C_E = \beta_0 < 0$ and $C_E = \alpha_0 = -\beta_0 > 0$ and thereby uniquely distinguish the pair of ICs Ψ_H and Ψ_b. The family of solutions of boundary-value problem (1.6), for which the ratio $(-b_1/b_2)$ has one and the same value, corresponds to this pair in the class of doubly continuously differentiable functions. Knowing b_1 and b_2, we determine c_b and c_H (at $C_g > 0$ from (3.1)) and obtain the unique solution of problem (1.6). When $\xi = 0$, we have

$$\frac{df_H}{d\xi} = \sqrt{2}c_H\beta_0, \quad \frac{df_b}{d\xi} = \sqrt{2}c_b\alpha_0 = (\sqrt{2}c_H\beta)$$

Consequently, boundary-value problem (1.6) can be replaced by the Cauchy problem for Eq. (1.6) with the initial conditions

$$f(0) = c_b^2(c_b \neq 0), \quad \frac{df}{d\xi}(0) = \sqrt{2}c_b\alpha_0, \quad \alpha_0 \in [0, \alpha^*)$$

After it has been solved, the solution of boundary-value problem (1.2)–(1.5) is determined as the function which is the inverse of

$$\zeta = \int_0^\Phi \frac{d\xi}{f(\xi)} \tag{3.5}$$

Its asymptotic behaviour as $|\zeta| \to \infty$ has the form

$$\Phi = b_1\left(\zeta + l_A^{(1)}\right)^m + \ldots, \quad \zeta \to +\infty; \quad l_A^{(1)} = -\int_0^\infty \left[\frac{1}{f(\xi)} - \frac{1}{mb_1^{\frac{1}{m}}\xi^{\frac{m-1}{m}}}\right]d\xi$$

$$\Phi = b_2\left(-\zeta + l_A^{(2)}\right)^m + \ldots, \quad \zeta \to -\infty; \quad l_A^{(2)} = -\int_{-\infty}^0 \left[\frac{1}{f(\xi)} - \frac{1}{m(-b_2)^{\frac{1}{m}}(-\xi)^{\frac{m-1}{m}}}\right]d\xi$$

The behaviour of the derivatives of the solution of (3.5) is determined using formulae (1.9). When $m = 1$, we have $d\Phi/d\zeta > 0$ and also $d^2\Phi/d\zeta^2 > 0$ if $c_b > 0$ and $d^2\Phi/d\zeta^2 < 0$ if $c_b < 0$.

Without the additional condition $d\Phi/d\zeta > 0$, the solution of boundary-value problem (1.2)–(1.5) ceases to be unique when $m > 1$.

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